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The E-characteristic polynomial of a tensor of dimension 2

Shenglong Hu *, Liqun Qi

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

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ABSTRACT

We show that the E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ of a tensor $\mathcal{T}$ of order $m \geq 3$ and dimension 2 is $\psi_{\mathcal{T}}(\lambda) = \det(S - \lambda T)$ with S a variant of the Sylvester matrix of the system $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$, and T a constant matrix that is only dependent on m. By exploring special structures of the matrices S and T, the coefficients of the E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ which make the computation of $\psi_{\mathcal{T}}(\lambda)$ efficient are obtained. On the basis of these, we prove that the leading coefficient of $\psi_{\mathcal{T}}(\lambda)$ is $(p_m^2 + q_m^2)^{\frac{m-2}{2}}$ when m is even and $-(p_m^2 + q_m^2)^{m-2}$ when m is odd, which strengthens Li, Qi and Zhang's theorem.

1. Introduction

By a tensor $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ of order m and dimension n with integers $m, n \geq 2$, we mean a collection of numbers $t_{i_1 \cdots i_m} \in \mathbb{C}$ for all $i_j \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. Given a vector $\mathbf{x} \in \mathbb{C}^n$, define an n-vector $\mathcal{T}\mathbf{x}^{m-1}$ with its ith element being $\sum_{i_2,\dots,i_m=1}^{n} t_{i i_2 \cdots i_m} \mathbf{x}_{i_2} \cdots \mathbf{x}_{i_m}$. E-eigenvalues of tensors were introduced by Qi in 2005 [1]. A number $\lambda \in \mathbb{C}$ is called an E-eigenvalue of $\mathcal{T}$ if it, together with a nonzero vector $\mathbf{x} \in \mathbb{C}^n$, satisfies

$$\begin{cases} \mathcal{T}\mathbf{x}^{m-1} = \lambda \mathbf{x}, \\ \mathbf{x}^T \mathbf{x} = 1. \end{cases}$$

$\mathbf{x}$ is then called the associated E-eigenvector of the E-eigenvalue λ, and $(\lambda, \mathbf{x})$ is called an eigenpair. The E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ is then defined as

$$\psi_{\mathcal{T}}(\lambda) := \begin{cases} \mathrm{Res}_{\mathbf{x}} \left(\mathcal{T}\mathbf{x}^{m-1} - \lambda (\mathbf{x}^T\mathbf{x})^{\frac{m-2}{2}} \mathbf{x} \right) & \text{when } m \text{ is even}, \\ \mathrm{Res}_{\mathbf{x},x_0} \begin{pmatrix} \mathcal{T}\mathbf{x}^{m-1} - \lambda x_0^{m-2} \mathbf{x} \\ \mathbf{x}^T\mathbf{x} - x_0^2 \end{pmatrix} & \text{when } m \text{ is odd}. \end{cases}$$

Here Res is the resultant of a system of polynomials [2–4]. The degree of the E-characteristic polynomial of a tensor was discussed in [5–7]. It was proven by Cartwright and Sturmfels [5] that $\deg(\psi_{\mathcal{T}}(\lambda)) = \frac{(m-1)^n - 1}{m-2}$ for generic even order tensors and $\deg(\psi_{\mathcal{T}}(\lambda)) = 2\frac{(m-1)^n - 1}{m-2}$ for generic odd order tensors.

More recently, Li et al. [8] showed that the constant term of $\psi_{\mathcal{T}}(\lambda)$ is $\mathrm{Res}_{\mathbf{x}}\left(\mathcal{T}\mathbf{x}^{m-1}\right)$ when m is even and $\left(\mathrm{Res}_{\mathbf{x}}\left(\mathcal{T}\mathbf{x}^{m-1}\right)\right)^2$ when m is odd. They also proved that the coefficients of the E-characteristic polynomial are invariant under the action of the group of orthogonal linear transformations. Moreover, they proved that when $n = 2$, the E-characteristic polynomial is the determinant of a matrix consisting of elements of $\mathcal{T}$ and λ. When a two-dimensional tensor has only finitely many equivalence classes of eigenpairs (two eigenpairs are considered to be in the same equivalence class if they are the same point in the weighted projective space $\mathbb{P}(m-2, \mathbf{1})$ [5]), the leading coefficient of $\psi_{\mathcal{T}}(\lambda)$ is proven [8, Theorem 7.4] to be

* Corresponding author. Tel.: +852 2766 4368; fax: +852 2362 9045.
E-mail addresses: Tim.Hu@connect.polyu.hk (S. Hu), maqilq@polyu.edu.hk (L. Qi).

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either $(p_m^2+q_m^2)^{\frac{m-2}{2}}$, when m is even, or $-(p_m^2+q_m^2)^{m-2}$, when m is odd, with p_m and q_m defined below (see (1) and (2)). In [5], Cartwright and Sturmfels showed that a tensor may have infinitely many equivalence classes of eigenpairs. Consequently, it is natural to ask *whether the theorem of Li et al.* [8, Theorem 7.4] *is still true in general.*

In this work, we give an affirmative answer to this question. The remainder of this work is organized as follows: some comments on notation are given in Section 2, the case for even order tensors is proven in Section 3, the case for odd order tensors is given in Section 4, and some final remarks are presented in the last section.

2. Preliminaries

We present, in this short section, the notation used in the subsequent analysis. Scalars are written as lower case letters $(\lambda, a, \dots)$, vectors are written as bold lower case letters $(\mathbf{x}, \mathbf{y}, \dots)$, the ith entry of a vector $\mathbf{x}$ is denoted by $\mathbf{x}_i$, matrices are written as italic capitals $(A, L, \dots)$, and tensors are written as calligraphic capitals $(\mathcal{T}, \mathcal{D}, \dots)$. For the convenience of the subsequent analysis, some notation is reserved as follows. Throughout the remaining work, the tensors referred to are two-dimensional tensors. Let the entries of tensor $\mathcal{T}$ be $t_{i_1\cdots i_m}$ with $i_j \in \{1, 2\}$ for $j \in \{1, \dots, m\}$. Define $\mathbf{b}_1 := t_{11\cdots 1}$, $\mathbf{b}_2 := \sum\{t_{1i_2\cdots i_m} | \{i_2, \dots, i_m\} = \{\underbrace{1, \dots, 1}_{m-2}, 2\}\}, \dots, \mathbf{b}_{m-1} := \sum\{t_{1i_2\cdots i_m} | \{i_2, \dots, i_m\} = \{1, \underbrace{2 \dots, 2}_{m-2}\}\}$, and $\mathbf{b}_m := t_{12\cdots 2}$; and $\mathbf{c}_1 := t_{21\cdots 1}$, $\mathbf{c}_2 := \sum\{t_{2i_2\cdots i_m} | \{i_2, \dots, i_m\} = \{1, \dots, 1, 2\}\}, \dots, \mathbf{c}_{m-1} := \sum\{t_{2i_2\cdots i_m} | \{i_2, \dots, i_m\} = \{1, 2 \dots, 2\}\}$, and $\mathbf{c}_m := t_{22\cdots 2}$. Define p_m and q_m respectively as

$$p_m := \mathbf{b}_1 - \mathbf{c}_2 - \mathbf{b}_3 + \mathbf{c}_4 + \mathbf{b}_5 - \mathbf{c}_6 - \mathbf{b}_7 + \mathbf{c}_8 + \mathbf{b}_9 - \mathbf{c}_{10} - \cdots, \tag{1}$$

$$q_m := \mathbf{c}_1 + \mathbf{b}_2 - \mathbf{c}_3 - \mathbf{b}_4 + \mathbf{c}_5 + \mathbf{b}_6 - \mathbf{c}_7 - \mathbf{b}_8 + \mathbf{c}_9 + \mathbf{b}_{10} - \cdots. \tag{2}$$

Definition 2.1. Given a vector $\mathbf{v} \in \mathbb{C}^k$, let $A(\mathbf{v}; m)$ denote the $m \times (k + m - 1)$ matrix generated as follows: its ith row begins with $i - 1$ zeros, $\mathbf{v}^T$ then follows, and it is finally complemented with $m - i$ zeros.

Given a matrix A, we denote by $A[i, j]$ its (i, j)th element, $A[i, :]$ its ith row, and $A[:, j]$ its jth column. The notation is generalized to $A[\mathbb{I}, \mathbb{J}]$, $A[\mathbb{I}, :]$ and $A[:, \mathbb{J}]$ for subsets $\mathbb{I}$ and $\mathbb{J}$ in a natural way. These are adopted from the notation in MatLab. Finally, $i : j$ means the set $\{i, i + 1, \dots, j\}$ when $i \le j$ and is vacuous otherwise.

3. The even order case

We show in this section that the leading coefficient of $\psi_{\mathcal{T}}(\lambda)$ is $(p_m^2 + q_m^2)^{\frac{m-2}{2}}$ for any even order tensor $\mathcal{T}$.

Proposition 3.1. *Suppose that $\mathcal{T}$ is a tensor of order $m = 2k + 2$ with $k \ge 1$; define vector $\mathbf{a} \in \mathbb{C}^{m-1}$ as*

$$\mathbf{a}^T := \left(\binom{k}{0}, 0, \binom{k}{1}, 0, \dots, \binom{k}{k}\right). \tag{3}$$

Then $\psi_{\mathcal{T}}(\lambda) = \det(S - \lambda T)$ with $S, T \in \mathbb{C}^{(2m-2)\times(2m-2)}$ defined respectively as

$$S := \begin{pmatrix} A(\mathbf{b}; m-1) \\ A(\mathbf{c}; m-1)[m-1, :] \\ A(\mathbf{b}; m-1)[2 : m-1, :] - A(\mathbf{c}; m-1)[1 : m-2, :] \end{pmatrix}, \tag{4}$$

and

$$T := \begin{pmatrix} A(\mathbf{a}; m) \\ 0_{m-2,2m-2} \end{pmatrix}. \tag{5}$$

Proof. Consider

$$F(\mathbf{x}, \lambda) := \begin{pmatrix} (\mathcal{T}\mathbf{x}^{m-1})_1 - \lambda(\mathbf{x}^T\mathbf{x})^{\frac{m-2}{2}}\mathbf{x}_1 \\ \mathbf{x}_2(\mathcal{T}\mathbf{x}^{m-1})_1 - \mathbf{x}_1(\mathcal{T}\mathbf{x}^{m-1})_2 \end{pmatrix} = \mathbf{0}. \tag{6}$$

It is easy to see that any nonzero solution of system

$$\mathcal{T}\mathbf{x}^{m-1} - \lambda(\mathbf{x}^T\mathbf{x})^{\frac{m-2}{2}}\mathbf{x} = \mathbf{0}$$

is a nonzero solution of the system (6). This implies that $\psi_{\mathcal{T}}(\lambda)$ is a factor of $\mathrm{Res}_{\mathbf{x}}F(\mathbf{x}, \lambda)$. Meanwhile, the only possible additional nonzero solution of (6) satisfies

$$\mathbf{x}_1 = 0, \quad \text{and} \quad (\mathcal{T}\mathbf{x}^{m-1})_1 = 0.$$

Since $(\mathcal{T}\mathbf{x}^{m-1})_1 = \sum_{i=1}^{m} \mathbf{b}_i \mathbf{x}_1^{m-i}\mathbf{x}_2^{i-1}$, $\mathbf{x}_1 = 0$, together with $\mathbf{x}_2 \neq 0$ and $(\mathcal{T}\mathbf{x}^{m-1})_1 = 0$, implies that $\mathbf{b}_m = t_{12\cdots 2} = 0$. Therefore, we conclude that

$$\mathrm{Res}_{\mathbf{x}} F(\mathbf{x},\lambda) = \mathbf{b}_m^r\,[\psi_{\mathcal{T}}(\lambda)]^s \tag{7}$$

for some integers r, s.

Now, we have

$$F(\mathbf{x},\lambda) = \begin{pmatrix} \sum_{j=0}^{k}\left[\left(\mathbf{b}_{2j+1} - \lambda\binom{k}{j}\right)\mathbf{x}_1^{m-2j-1}\mathbf{x}_2^{2j} + \mathbf{b}_{2j+2}\mathbf{x}_1^{m-2j-2}\mathbf{x}_2^{2j+1}\right] \\ -\mathbf{c}_1\mathbf{x}_1^m + \sum_{i=1}^{m-1}(\mathbf{b}_j - \mathbf{c}_{j+1})\mathbf{x}_1^{m-i}\mathbf{x}_2^i + \mathbf{b}_m\mathbf{x}_2^m \end{pmatrix}.$$

By the Sylvester formula [3,2], $\mathrm{Res}_{\mathbf{x}} F(\mathbf{x},\lambda)$ is the determinant of the following $(2m-1)\times(2m-1)$ matrix:

$$\begin{pmatrix} A(\mathbf{b}; m) - \lambda A(\mathbf{a}; m+1)[1:m,:] \\ A(\mathbf{b}; m)[2:m,:] - A(\mathbf{c}; m)[1:m-1,:] \end{pmatrix}$$
$$= \begin{pmatrix} A(\mathbf{b}; m) \\ A(\mathbf{b}; m)[2:m,:] - A(\mathbf{c}; m)[1:m-1,:] \end{pmatrix} - \lambda \begin{pmatrix} A(\mathbf{a}; m+1)[1:m,:] \\ 0_{m-1,2m-1} \end{pmatrix}.$$

Subtract the last row from the mth row of the above matrix; we get the following matrix:

$$\begin{pmatrix} A(\mathbf{b}; m)[1:m-1,:] \\ A(\mathbf{c}; m)[m-1,:] \\ A(\mathbf{b}; m)[2:m,:] - A(\mathbf{c}; m)[1:m-1,:] \end{pmatrix} - \lambda \begin{pmatrix} A(\mathbf{a}; m+1)[1:m,:] \\ 0_{m-1,2m-1} \end{pmatrix}.$$

Observe that all the elements of the last column of the matrix above are zero except the bottom element, which is $\mathbf{b}_m$. Consequently, $\mathrm{Res}_{\mathbf{x}} F(\mathbf{x},\lambda) = \mathbf{b}_m \det(S - \lambda T)$ with S and T being defined by (4) and (5) respectively. Now, $\det(S-\lambda T)$ is a homogeneous polynomial in the variables $t_{i_1\cdots i_m}$ and λ of degree $2(m-1)$. By the properties of the resultant [4,3,2], $\psi_{\mathcal{T}}(\lambda) = \mathrm{Res}_{\mathbf{x}}\left(\mathcal{T}\mathbf{x}^{m-1} - \lambda(\mathbf{x}^T\mathbf{x})^{\frac{m-2}{2}}\mathbf{x}\right)$ is a homogeneous polynomial in the variables $t_{i_1\cdots i_m}$ and λ of degree $2(m-1)$. These results, together with (7), imply that $r = s = 1$. Consequently, $\psi_{\mathcal{T}}(\lambda) = \det(S - \lambda T)$. $\square$

Remark 3.1. A tensor $\mathcal{T}$ is called regular if and only if system $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}, \mathbf{x}^T\mathbf{x} = 0$ has only the trivial solution [1]. By Definition 2.1, Proposition 3.1 follows from [8, Theorem 4.1] directly for regular tensors. Actually, from the proof of [8, Theorem 4.1], we see that regularity is redundant and the result is true for all tensors. We include the proof here also for completeness.

Given two matrices $A, B \in \mathbb{C}^{n\times n}$, define $\det(A,B,k)$ as the sum of the determinants of the matrices C obtained by replacing k rows of A by the corresponding rows of B. Obviously, $\det(A,B,k) = \det(B,A,n-k)$, $\det(A,B,0) = \det(A)$, and $\det(A,B,n) = \det(B)$.

Lemma 3.1. *Given two matrices $A, B \in \mathbb{C}^{n\times n}$, we have that $\det(A+B) = \sum_{k=0}^{n}\det(A,B,k)$.*

Proof. It follows from [9, Lemma 2] immediately. $\square$

Now, we have the following result.

Proposition 3.2. *Suppose that $\mathcal{T}$ is a tensor of order $m = 2k+2$ with $k \geq 1$; the matrices S and T are defined as those in Proposition 3.1. Then,*

$$\psi_{\mathcal{T}}(\lambda) = \sum_{k=0}^{m}(-\lambda)^k \det(S,T,k). \tag{8}$$

In particular,

(i) *the constant term of $\psi_{\mathcal{T}}(\lambda)$ is $\det(S) = Res_{\mathbf{x}}(\mathcal{T}\mathbf{x}^{m-1})$, the resultant of $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$, and*
(ii) *the leading coefficient of $\psi_{\mathcal{T}}(\lambda)$ is $\det(L)$ with*

$$L := \begin{pmatrix} -A(\mathbf{a}; m) \\ A(\mathbf{b}; m-1)[2:m-1,:] - A(\mathbf{c}; m-1)[1:m-2,:] \end{pmatrix}. \tag{9}$$

Proof. The formula (8) follows from Lemma 3.1 and the fact that only m rows of T are not identically zero.

For the result (i) on the constant term, note that

$$\begin{aligned}\det(S) &= \det\begin{pmatrix} A(\mathbf{b}; m-1) \\ A(\mathbf{c}; m-1)[m-1, :] \\ A(\mathbf{b}; m-1)[2 : m-1, :] - A(\mathbf{c}; m-1)[1 : m-2, :]\end{pmatrix} \\ &= \det\begin{pmatrix} A(\mathbf{b}; m-1) \\ A(\mathbf{c}; m-1)[m-1, :] \\ -A(\mathbf{c}; m-1)[1 : m-2, :]\end{pmatrix} \\ &= \det\begin{pmatrix} A(\mathbf{b}; m-1) \\ A(\mathbf{c}; m-1)\end{pmatrix} \\ &= \mathrm{Res}_{\mathbf{x}}(\mathcal{T}\mathbf{x}^{m-1}),\end{aligned}$$

where the second equality follows from the fact that the determinant is invariant under elementary row transformations, the third from the fact that $m \geq 4$ is even, and the last one from the definition of the resultant.

Lastly, the result (ii) for formula (9) follows from the definition of $\det(S, T, m)$. □

From Proposition 3.2, we see that the coefficient of λ^k in the E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ is $(-1)^k \det(S, T, k)$ for all $k \in \{0, \ldots, m\}$. As the coefficients are invariants [8], such explicit formulas are useful for further investigation of $\psi_{\mathcal{T}}(\lambda)$.

The following observations are crucial for the main theorem.

Lemma 3.2. *Suppose that $\mathcal{T}$ is a tensor of order $m = 2k+2$ with $k \geq 1$, and matrix L is defined by (9). We have that $\det(L) = 0$ if and only if the following system has a solution:*

$$\begin{cases}(t^2+1)^{\frac{m-2}{2}} = 0, \\ \displaystyle\sum_{i=1}^{m} \mathbf{c}_i t^{m-i+1} - \sum_{i=1}^{m} \mathbf{b}_i t^{m-i} = 0.\end{cases} \tag{10}$$

Proof. Write out the matrix L; the result follows from directly from the resultant theory. Actually, matrix $-L$ is the Sylvester matrix of the system (10). □

Lemma 3.3. *Suppose that $\mathcal{T}$ is a tensor of order $m = 2k+2$ with $k \geq 1$, and matrix L is defined by (9). We have that $\det(L) = 0$ if and only if $p_m^2 + q_m^2 = 0$.*

Proof. Note that the system (10) has a solution if and only if

$$\sum_{i=1}^{m} \mathbf{c}_i (\pm i)^{m-i+1} - \sum_{i=1}^{m} \mathbf{b}_i (\pm i)^{m-i} = 0, \tag{11}$$

since the only possible solutions are $t = \pm i$. Then, by the definitions of p_m and q_m (see (1) and (2)), (11) is equivalent to

$$q_m \pm i p_m = 0$$

which is further equivalent to $p_m^2 + q_m^2 = 0$. Now, the result follows from Lemma 3.2. □

We now prove the main theorem.

Theorem 3.1. *Suppose that $\mathcal{T}$ is a tensor of order $m = 2k+2$ with $k \geq 1$. Then, the leading coefficient of its E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ is $(p_m^2 + q_m^2)^{\frac{m-2}{2}}$.*

Proof. Denote by $\mathbb{C}[\mathcal{T}]$ the polynomial ring with indeterminant $\mathcal{T}$ and coefficients in the field $\mathbb{C}$ of complex numbers. For any $f \in \mathbb{C}[\mathcal{T}]$, denote by $\mathbb{V}(f)$ the variety of f and by $\mathbb{I}(\mathbb{V}(f))$ the ideal generated by $\mathbb{V}(f)$ [2,3]. Then, the two varieties

$$\begin{aligned} W &:= \mathbb{V}(\det(L)) = \{\mathcal{T} \mid \det(L) = 0\}, \quad \text{and} \\ V &:= \mathbb{V}(p_m^2 + q_m^2) = \{\mathcal{T} \mid p_m^2 + q_m^2 = 0\}\end{aligned}$$

are equal by Lemma 3.3. Consequently, $\det(L) \in \mathbb{I}(W) = \mathbb{I}(V) = \mathbb{I}(\mathbb{V}(\langle p_m^2 + q_m^2\rangle))$. Hence, $(\det(L))^k = (p_m^2 + q_m^2) \cdot p$ for some integer k and $p \in \mathbb{C}[\mathcal{T}]$ by Hilbert's Nullstellensatz [2, Theorem 4.2]. Now, by (1) and (2), $(p_m + iq_m)(p_m - iq_m)$ is an irreducible decomposition of $p_m^2 + q_m^2$. So, $\det(L) = (p_m + iq_m)^s (p_m - iq_m)^t \cdot w$ for some integers s and t, and $w \in \mathbb{C}[\mathcal{T}]$ which is coprime with both $p_m + iq_m$ and $p_m - iq_m$. Similarly, $(p_m^2 + q_m^2)^l = (\det(L)) \cdot r$ for some integer l and $r \in \mathbb{C}[\mathcal{T}]$. Consequently, $w \in \mathbb{C} \setminus \{0\}$.

Now, the fact that $\det(L) \in \mathbb{Z}[\mathcal{T}]$ implies that $s = t$, while [8, Theorem 7.4] implies that $w = 1$. These, together with the facts that $\psi_{\mathcal{T}}(\lambda) \in \mathbb{C}[\mathcal{T}, \lambda]$ is a homogeneous polynomial of degree $2(m-1)$ [3, Theorem 3.2.3], and $\deg(\psi_{\mathcal{T}}(\lambda)) = m$ when it is considered as an element in $\mathbb{C}[\lambda]$ by Propositions 3.2 and 3.2(ii), imply that the leading coefficient of the E-characteristic polynomial $\psi_{\mathcal{T}}(\lambda)$ is $(p_m^2 + q_m^2)^{\frac{m-2}{2}}$. □

4. The odd order case

On the basis of [8, Theorem 4.2] and similar analysis in Section 3, we can solve the odd case problem. The proof for [8, Theorem 4.2] is similar to that for Proposition 3.1. We omit most of the similar proofs and present the main result in the following theorem. To this end, define a vector $\mathbf{m}$ through $\mathbf{m}_i := \sum_{1\le j,k\le m, j+k=i+1} \mathbf{b}_j\mathbf{c}_k$ for all $i \in \{1, \dots, 2m-1\}$.

Theorem 4.1. *Suppose that $\mathcal{T}$ is a tensor of order $m = k+2$ with $k \ge 1$ being odd, and define vector $\mathbf{a} \in \mathbb{C}^{2m-3}$ as that in Proposition 3.1. Then:*

(i) $\psi_{\mathcal{T}}(\lambda) = \det(S - \lambda^2 T)$ *with* $S, T \in \mathbb{C}^{(3m-4)\times(3m-4)}$ *respectively defined as*

$$S := \begin{pmatrix} A(\mathbf{m}; m)[1, \mathbb{J}] + \mathbf{b}_1(A(\mathbf{b}; 2m)[2, \mathbb{J}] - A(\mathbf{c}; 2m)[1, \mathbb{J}]) \\ A(\mathbf{m}; m-1)[1 : m-2, :] \\ A(\mathbf{m}; m)[m, \mathbb{J}] - \mathbf{c}_m(A(\mathbf{b}; 2m)[2m-1, \mathbb{J}] - A(\mathbf{c}; 2m)[2m-2, \mathbb{J}]) \\ A(\mathbf{b}; 2m-3)[2 : 2m-3, :] - A(\mathbf{c}; 2m-3)[1 : 2m-4, :] \end{pmatrix}$$

with $\mathbb{J} = 2 : 3m-3$, *and*

$$T := \begin{pmatrix} A(\mathbf{a}; m) \\ 0 \end{pmatrix}.$$

Consequently,

$$\psi_{\mathcal{T}}(\lambda) = \sum_{k=0}^{m} (-\lambda^2)^k \det(S, T, k). \tag{12}$$

(ii) *The constant term of* $\psi_{\mathcal{T}}(\lambda)$ *is* $\det(S) = \left(Res_{\mathbf{x}}(\mathcal{T}\mathbf{x}^{m-1})\right)^2$, *the square of the resultant of* $\mathcal{T}\mathbf{x}^{m-1} = \mathbf{0}$.

(iii) *The leading coefficient of* $\psi_{\mathcal{T}}(\lambda)$ *is* $\det(L)$ *with*

$$L := \begin{pmatrix} A(\mathbf{a}; m) \\ A(\mathbf{b}; 2m-3)[2 : 2m-3, :] - A(\mathbf{c}; 2m-4) \end{pmatrix}.$$

Moreover, it is equal to $-(p_m^2 + q_m^2)^{m-2}$.

Proof. The proof for the result (i) is similar to that for Proposition 3.1. The difference is that instead of investigating $F(\mathbf{x}, \lambda)$ defined by (6), we investigate

$$G(\mathbf{x}, \lambda) := \begin{pmatrix} (\mathcal{T}\mathbf{x}^{m-1})_1(\mathcal{T}\mathbf{x}^{m-1})_2 - \lambda^2(\mathbf{x}^T\mathbf{x})^{\frac{2m-4}{2}}\mathbf{x}_1\mathbf{x}_2 \\ \mathbf{x}_2(\mathcal{T}\mathbf{x}^{m-1})_1 - \mathbf{x}_1(\mathcal{T}\mathbf{x}^{m-1})_2 \end{pmatrix} = \mathbf{0}.$$

Consequently, we now only give the proof for $\det(S) = \left(\mathrm{Res}_{\mathbf{x}}(\mathcal{T}\mathbf{x}^{m-1})\right)^2$. The others are similar to those in Section 3. Note that

$$\mathbf{b}_m\mathbf{c}_1 \det(S) = \det\begin{pmatrix} A(\mathbf{m}; m) \\ A(\mathbf{b}; 2m-1)[2 : 2m-1, :] - A(\mathbf{c}; 2m-1)[1 : 2m-2, :] \end{pmatrix}$$

by elementary row transformations. Now, by direct computation and the resultant theory, we can get that the latter determinant is actually the resultant of

$$\begin{cases} \left(\sum_{i=1}^{m} \mathbf{c}_i t^{m-i}\right)\left(\sum_{i=1}^{m} \mathbf{b}_i t^{m-i}\right) = 0, \\ \sum_{i=1}^{m} \mathbf{c}_i t^{m-i+1} - \sum_{i=1}^{m} \mathbf{b}_i t^{m-i} = 0. \end{cases} \tag{13}$$

Define $f(t) := \sum_{i=1}^{m} \mathbf{c}_i t^{m-i} \in \mathbb{C}[t]$, and $g(t) := \sum_{i=1}^{m} \mathbf{b}_i t^{m-i} \in \mathbb{C}[t]$. Then, we can get that

$$\begin{aligned} \mathbf{b}_m\mathbf{c}_1 \det(S) &= \mathrm{Res}(fg, tf - g) \\ &= \mathrm{Res}(f, tf - g)\mathrm{Res}(g, tf - g) \\ &= \det\begin{pmatrix} A(\mathbf{c}; m) \\ A(\mathbf{c}; m)[1 : m-1, :] - A(\mathbf{b}; m)[2 : m, :] \end{pmatrix} \cdot \det\begin{pmatrix} A(\mathbf{b}; m) \\ A(\mathbf{c}; m)[1 : m-1, :] - A(\mathbf{b}; m)[2 : m, :] \end{pmatrix} \\ &= \det\begin{pmatrix} A(\mathbf{c}; m) \\ -A(\mathbf{b}; m)[2 : m, :] \end{pmatrix} \cdot \det\begin{pmatrix} A(\mathbf{b}; m) \\ A(\mathbf{c}; m)[1 : m-1, :] \end{pmatrix} \\ &= \mathbf{c}_1 \det\begin{pmatrix} A(\mathbf{c}; m-1) \\ -A(\mathbf{b}; m-1) \end{pmatrix} \cdot (-1)^{m+1}\mathbf{b}_m \det\begin{pmatrix} A(\mathbf{b}; m-1) \\ A(\mathbf{c}; m-1) \end{pmatrix} \end{aligned}$$

$$= (-1)^{2m}\mathbf{c}_1\mathbf{b}_m\,(\mathrm{Res}(f,g))^2$$
$$= \mathbf{c}_1\mathbf{b}_m\left(\mathrm{Res}_{\mathbf{x}}(\mathcal{T}\mathbf{x}^{m-1})\right)^2.$$

Here, the second equality follows from (13), [3, Theorem 3.3.2(b)] and a homogenization procedure; the fourth from the fact that the determinant is invariant under elementary row transformations; the fifth from the structure of the matrices in the fourth equality; the sixth from the definition of the resultant; and the last one from the definitions of f, g and the resultant. □

Remark 4.1. Theorems 3.1 and 4.1 strengthen Li, Qi and Zhang's Theorem [8, Theorem 7.4]. The proof also gives an alternative proof for [8, Theorem 3.3] when $n = 2$.

5. Final remarks

Though Li et al. [8] give the formula for $\psi_{\mathcal{T}}(\lambda)$ as the determinant of a matrix consisting of entries of $\mathcal{T}$ and λ, the effort required for direct computation is great, especially when m is large, since it involves the determinant of a symbolical matrix (see [8, Theorems 4.1 and 4.2]). From this perspective, the reformulations (8) and (12) have their merits in terms of (8).

At the end of this work, we remark on the computation of $\psi_{\mathcal{T}}(\lambda)$ through our reformulations for $\psi_{\mathcal{T}}(\lambda)$. In particular, we give formulas for the inverse of the leading principal submatrix of T defined as (5). This, together with Schur complement theory, is useful for computing the coefficients of $\psi_{\mathcal{T}}(\lambda)$.

For general $T \in \mathbb{C}^{(2m-2)\times(2m-2)}$, we can partition as follows:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{14}$$

with $T_{11} \in \mathbb{C}^{m\times m}$, $T_{12} \in \mathbb{C}^{m\times(m-2)}$, $T_{21} = 0 \in \mathbb{C}^{(m-2)\times m}$ and $T_{22} = 0 \in \mathbb{C}^{(m-2)\times(m-2)}$.

Lemma 5.1. *Given a vector $\mathbf{v} \in \mathbb{C}^k$ with $\mathbf{v}_1 \neq 0$, let B be the rth-order leading principal submatrix of $A(\mathbf{v}; m)$ with $r \leq m$, i.e., $B = A(\mathbf{v}; m)[1 : r, 1 : r]$. Then, B is invertible and its inverse is the rth-order leading principal submatrix of $A(\mathbf{u}; m)$ for some $\mathbf{u} \in \mathbb{C}^k$ with $\mathbf{u}_1 = \frac{1}{\mathbf{v}_1}$.*

Proof. The invertibility of B follows from the fact that it is an upper triangular matrix with the diagonal elements being $\mathbf{v}_1$'s. Now, suppose that $K = B^{-1}$ is its inverse. We know from matrix theory that K is also an upper triangular matrix with the diagonal elements being $\frac{1}{\mathbf{v}_1}$'s. From the equations of $0 = KB[1, i]$ for $i \geq 2$, we can obtain the first row of the matrix K which is denoted as $\mathbf{u} \in \mathbb{C}^k$ after complementation with zeros. Now, it is a trivial task to show that $B\,(A(\mathbf{u}; m)[1 : r, 1; r]) = (A(\mathbf{u}; m)[1 : r, 1 : r])\,B = I$, the identity matrix. By the uniqueness of K, we conclude that $K = A(\mathbf{u}; m)[1 : r, 1; r]$. □

Here is the result that we promised.

Proposition 5.1. *For any $m \geq 2$, and T_{11} defined by (14) with T being defined by (5), we have that T_{11} is invertible and*

$$T_{11}^{-1} = A(\mathbf{d}; m)[:, 1 : m]$$

with $\mathbf{d} \in \mathbb{C}^m$ defined recursively as $d_1 = 1$, $d_2 = 0$, and

$$\mathbf{d}_{2j+1} = -\sum_{i=1}^{j-1}\mathbf{a}_{2i+1}\mathbf{d}_{2j+1-2i} = -\sum_{i=1}^{j-1}\binom{k}{i}\mathbf{d}_{2j+1-2i},\ \mathbf{d}_{2j+2} = 0, \quad \text{for } j \geq 1.$$

Proof. It follows from Lemma 5.1, (3) and (5) directly. □

In this work, we complemented Li, Qi and Zhang's theorem [8, Theorem 7.4] and gave alternative reformulations of the E-characteristic polynomials of tensors of dimension 2. Such reformulations have their own merits and the explicit structures of the coefficients of the E-characteristic polynomials have potential value in investigation of the invariants of tensors of dimension 2.

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